From the morphism between QFTs to topological Wick rotation I

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References: K.-Zheng:1705.01087,1905.04924,1912.01760, 2011.02859

The proposal of sandwich construction or SymTFT or topological holography (within high energy community): Gaiotto-Kulp:2008.05960, Bhardwaj-Lee-Tachikawa:2009.10099, Freed-Moore-Teleman:2209.07471, Apruzzi-Bonetti-Etxebarria-Hosseini-Schäfer-Nameki:2112.02092

$$X^n := \mathrm{B}^n_{\mathrm{sym}} \boxtimes_{\mathsf{C}^{n+1}} \mathrm{B}^n_{\mathrm{dyn}} := \left\{ \begin{array}{c} \text{gapped boundary } \mathrm{B}^n_{\mathrm{sym}} \\ \hline \text{SymTFT } \mathsf{Z}(\mathrm{B}_{\mathrm{sym}})^{n+1} \\ \hline \text{gapless boundary } \mathrm{B}^n_{\mathrm{dyn}} \end{array} \right.$$

1. gapped boundary $\mathrm{B}^n_{\mathrm{sym}}$ encodes the information of non-invertible symmetries;
2. gapless boundary $\mathrm{B}^n_{\mathrm{dyn}}$ encodes the information of dynamical data.

In these three talks, I will show that this idea of SymTFT is a consequence of boundary-bulk relation, i.e., "bulk=Z(boundary)" K.-Zheng:1502.01690, and the mathematical theory of gapped/gapless boundaries of 2+1D topological orders developed in K.-Zheng:1705.01087 (more details in 1905.04924,1912.01760). This theory provides a precise meaning of above sandwich expansion and a powerful way to compute the sandwich. Therefore, my talk can also be viewed as an introduction to SymTFT or the famous slogan "Symmetry is a 1-dimensional higher topological order".

Some psychological preparation are needed before we start.

1. I will emphasize the physical intuitions instead of mathematical precision or rigorousness, which should be viewed only as supporting evidence of our physical intuitions.
2. I will show how formal physical/geometric intuitions, which seems to be abstract and tautological nonsense, gradually lead us to non-trivial physical results and even powerful tools for computation.
3. These physical intuitions and results gradually grow into a robust theoretical framework, which I called global quantum many-body theory or global quantum field theory.

Maybe a better title of my lectures is: How to turn abstract nonsense into deep principles, powerful tools, concrete calculation and robust theoretical framework?

Disclaimer: In my talk, we ignore the microscopic origins (or realizations) of a quantum phase/liquid but focus only on its macroscopic properties. This is possible because the notion of a quantum phase is 'essentially' a macroscopic notion defined in the long wave length limit.

An evolution from a concrete approach to an abstract approach is actually a evolution from the study of individual defects to that of all defects.

- There is a long history (no later than 1930s) of studying (topological/conformal/$\cdots$) defects in quantum many-body systems or QFTs.
- In the early age (1930s-1970s), the study of (topological) defects focuses on the physical realization of an individual defect and its topological properties.
- In the modern age (1980s-now), an important discovery is the global algebraic structures of the collection of all topological defects, such as modular tensor categories of anyons or fusion n-categories $\mathcal{A}$ of all topological defects of all codimensions. Moore-Seiberg:1989, Kitaev:quant-ph/9707021, cond-mat/0506438, Levin-Wen:cond-mat/0404617, Frölich-Fuchs-Runkel-Schweigert:hep-th/0607247, Kitaev-K.:1104.5047, K.-Wen:1405.5858, K.-Wen-Zheng:1502.01690, Lan-K.-Wen:1602.05946, Johnson-Freyd:2003.06663, K.-Zheng:2011.02859,2107.03858
  Objects or 0-morphisms in $\mathcal{A}$ are topological defects of codimension 1; k-morphisms in $\mathcal{A}$ are topological defects of codimension $k+1$. Defects can be fused and braided.

The main goal of global quantum many-body theory is not only to study each global algebraic structure but also to find the rich web of interrelation among them.

From the morphism of QFTs to the principle of ‘bulk=Z(boundary)’

Question: How to define a morphism between two QFT's?

A well known answer: A usual definition of a morphism between two QFT's is a domain wall between two QFT's. A domain wall provides a physical realization of the mathematical notion of a bimodule because the domain wall is naturally equipped with the two-side action of the operators in two QFT's.



However, such a definition of a morphism (as a bimodule) is less fundamental because it does not preserve the algebraic structures of a QFT. As a consequence, such a definition of a morphism (as a bimodule) only distinguishe QFT's up to Morita equivalences.

In mathematics, there is a more fundamental or natural notion of a homomorphism between two algebraic objects, i.e. a map preserving the algebraic structure.

1. A group homomorphism $f : G \to H$: $f(g_1 g_2) = f(g_1) f(g_2)$ for $g_1, g_2 \in G$.
2. An algebra homomorphism $f : A \to B$ between two $\mathbb{C}$-linear algebras is a $\mathbb{C}$-linear map such that $f(ab) = f(a)f(b)$.

Question: What is the physical realization of the notion of an algebra homomorphism?

Or equivalently:

Question: How to map a quantum many-body system (or QFT) to another such that algebraic structures of operators or observables are preserved?

Before we introduce this notion, we need some preparation. The term "topological order" in the following a few slides will be replaced eventually by (gapped/gapless) "quantum liquid" or certain 'nice' QFT, a notion which is, however, harder to define. Therefore, for convenience, we restrict our discussion to the familiar notion of a topological orders at first. We will come back to the more general setting at the end of this talk.

Definition (K.-Wen:1405.5858)

A topological order is called anomaly-free if it can be realized by a lattice model in the same dimension, and is called anomalous otherwise.

Examples: A gapped boundary phase of the trivial n+1D topological order is an anomaly-free nD topological order.

Unique Bulk Hypothesis/Principle [K.-Wen:1405.5858]

A (potentially anomalous) nD topological order A^n has a unique $n{+}1$D anomaly-free topological order as its *bulk*, denoted by $\mathrm{Z}(\mathrm{A}^n) = \mathrm{Z}(\mathrm{A})^{n+1}$.

Remark: An anomalous topological order must be realizable as a defect in a higher (but still finite) dimensional lattice model as illustrated below. Otherwise, it is safe to say that such a topological order does not exist. But such realizations are almost never unique.

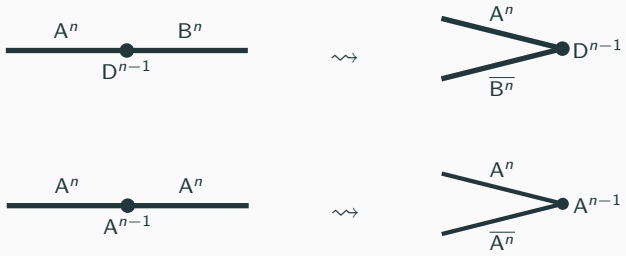


However, after the dimensional reduction, we obtain the unique *bulk* of A^n. Importantly, A^n should be understood as a boundary phase, which includes a neighborhood of the boundary by definition.

Example: If D^{n-1} is a domain wall between A^n and B^n, then $\mathrm{Z}(\mathrm{D})^n = \mathrm{A}^n \boxtimes \overline{\mathrm{B}}^n$.



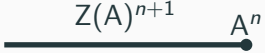
Example: If D^{n-1} is a domain wall between A^n and B^n, then $\mathrm{Z}(\mathrm{D})^n = \mathrm{A}^n \boxtimes \overline{\mathrm{B}}^n$.



where A^{n-1} denotes the A^n restricting to the trivial domain wall of codimension 1.

Example: $\mathbb{1}^n$ = the trivial nD topological order. We have

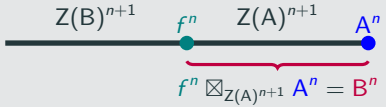
- $\mathrm{Z}(\mathbb{1})^{n+1} = \mathbb{1}^{n+1}$;
- A^n is anomaly-free if and only if $\mathrm{Z}(\mathrm{A})^{n+1} = \mathbb{1}^{n+1}$;
- $\mathrm{Z}(\mathrm{Z}(\mathrm{A}))^{n+2} = \mathbb{1}^{n+2}$.



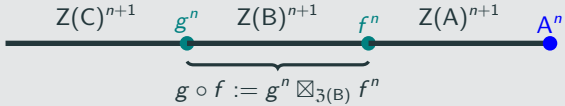
Remark: The statement of "the *bulk* of a *bulk* is trivial" is somewhat dual to that of "the boundary of a boundary is empty", which inspired homology theory. Therefore, we expect that "the *bulk* of a *bulk* is trivial" should lead us to a non-trivial but yet unknown "cohomology theory".

Definition (K.-Wen-Zheng:1502.01690, 1702.00673)

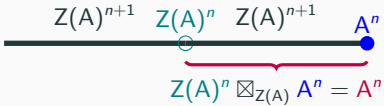
A morphism $f : \mathrm{A}^n \to \mathrm{B}^n$ between two topological orders A^n and B^n is a gapped domain wall f^n between $\mathrm{Z(A)}^{n+1}$ and $\mathrm{Z(B)}^{n+1}$ such that $f^n \boxtimes_{\mathrm{Z(A)}^{n+1}} \mathrm{A}^n = \mathrm{B}^n$.



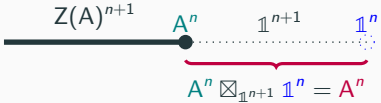
The composition of two morphisms $f : \mathrm{A}^n \to \mathrm{B}^n$ and $g : \mathrm{B}^n \to \mathrm{C}^n$ is defined and obviously associative.



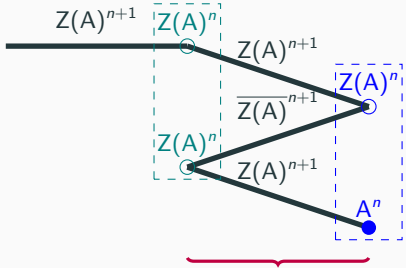
Examples 1: $\mathrm{A}^n \xrightarrow{\mathrm{id}_{\mathrm{A}}} \mathrm{A}^n$ is defined by the trivial domain wall $\mathrm{Z}(\mathrm{A})^n$ in $\mathrm{Z}(\mathrm{A})^{n+1}$.



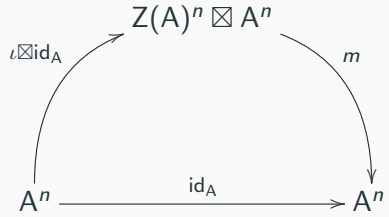
Examples 2: Let $\mathbb{1}^n$ be the trivial nD topological order. There is a canonical morphism $\iota : \mathbb{1}^n \to \mathrm{A}^n$ defined by:

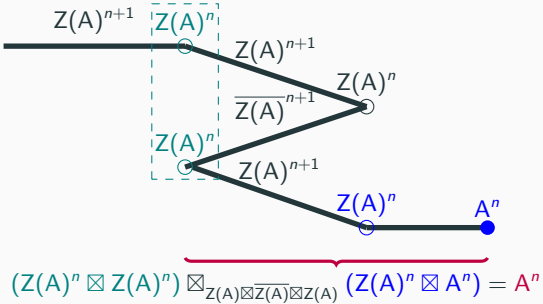


Examples 3: There is a canonical morphism $\mathrm{Z}(\mathrm{A})^n \boxtimes \mathrm{A}^n \xrightarrow{m} \mathrm{A}^n$ defined as follows

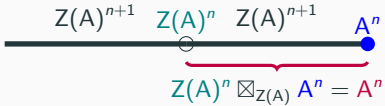


These three morphisms: id_A, ι, m form a commutative diagram.



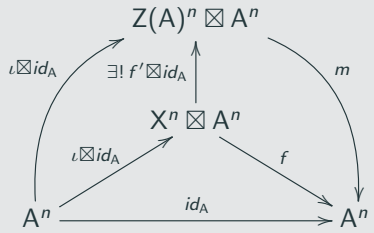


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Theorem (K.-Wen-Zheng:1502.01690, 1702.00673)

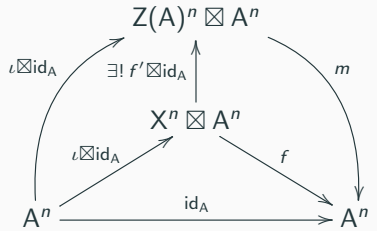
The pair $(\mathrm{Z}(\mathrm{A})^n, m)$ *satisfies the universal property of center. That is, if* X^n *is an nD topological order equipped with a morphism* $f : \mathrm{X}^n \boxtimes \mathrm{A}^n \to \mathrm{A}^n$ *rendering the same triangle commutative, then there is a unique morphism* $f' : \mathrm{X}^n \to \mathrm{Z}(\mathrm{A})^n$ *such that the following diagram is commutative:*



Remark: This theorem simply says that the *bulk* $\mathrm{Z}(\mathrm{A})^n$ is the center of the boundary A^n.

This universal property works for all the well-known notions of centers.

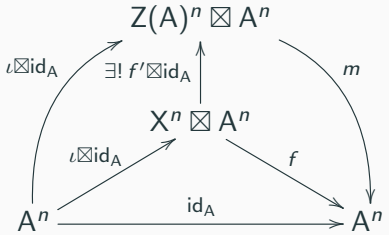
1. When A is a group and m is a group homomorphism, it recovers the center of a group $\mathfrak{Z}(\mathrm{A}) = \{z \in \mathrm{A} | zg = gz, \forall g \in \mathrm{A}\}$.
2. When A is an algebra and m is an algebraic homomorphism, it recovers the usual center of an algebra $\mathfrak{Z}(\mathrm{A}) = \{z \in \mathrm{A} | za = az, \forall a \in \mathrm{A}\}$.
3. When A is a fusion category and m is a monoidal functor, it recovers the Drinfeld center.
4. When A is a braided fusion category and m is a braided monoidal functor, it recovers the Müger center.



Warning:

We have obtained a theorem "bulk=Z(boundary)", which only makes sense in the category of (potentially anomalous) topological orders with our chosen definition of a morphism between objects.

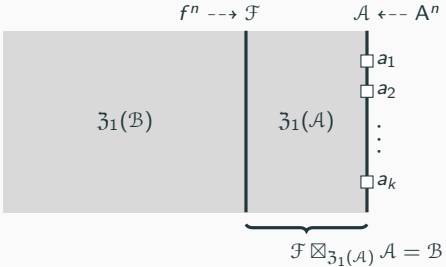
However, even if we know that a mathematical characterization of a boundary nD topological order X^n is given by a fusion n-category $\mathcal{X}$, we can not conclude that the mathematical characterization of the bulk is given by the the monoidal center of $\mathcal{X}$, i.e., $\mathfrak{Z}_1(\mathcal{X})$, because it is not yet known whether the morphism between two topological orders are compatible with the usual notion of a monoidal functor between two fusion n-categories.



Morphisms are compatible with monoidal functors



This definition of a morphism
coincides with the mathematical notion of
a monoidal functor between two fusion n-categories
K.-Zheng:1507.00503, 2107.03858



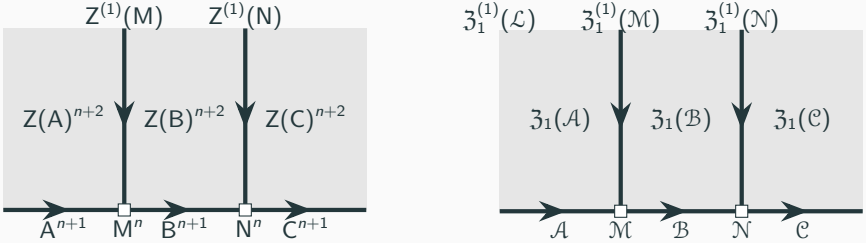
The idea of 'sandwich construction' for the definition of a morphism:

1. Any physical process that probes the interior of the phase A^n can alter the algebraic structure of the local operators or local observables.
2. In order to preserve the algebraic structure, only reasonable thing to do is to 'massage' the system A^n from 'outside'.

Some remarks on the history of 'bulk=Z(boundary)' in QFTs.

- 1+1D TQFT/RCFT:
  1. open-closed TQFT, open algebra $= M_{n\times n}$ and closed algebra $= \mathbb{C} = Z(M_{n\times n})$. Moore:2001
  2. a slightly weaker result in 1+1D topological conformal field theory (TCFT): closed TCFT can be HH*(open TCFT) Kontsevich:1990s, Costello:math/0412149
  3. 1+1D open-closed RCFT, closed CFT = the 'full center' of open CFT
     Fjelstad-Fuchs-Runkel-Schweigert:hep-th/0503194, K.-Runkel:0708.189, Davydov:0908.1044
- Topological orders or TQFTs:
  1. In Turaev-Viro-Barrett-Westbury's state-sum construction of 2+1D TQFTs, in which the spherical fusion category appeared as an initial data of the construction, boundaries of the TQFT were not studied at all!
  2. For 2+1D topological orders with a gapped boundary, bulk MTC = Drinfeld center of the boundary fusion category Kitaev-K.:1104.5047, K.:1307.8244
  3. Topological orders (tautologically generalizable to QFT's) in all dimensions K.-Wen-Zheng:1502.01690
  4. It leads to a unified theory of gapped and gapless boundaries of topological orders. K.-Zheng:1705.01087
  5. Generalizing 2+1D Turaev-Viro TQFTs to include boundaries Freed-Teleman:2006.10200

‘Bulk = Z(boundary)’ is functorial

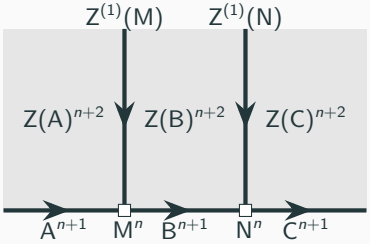


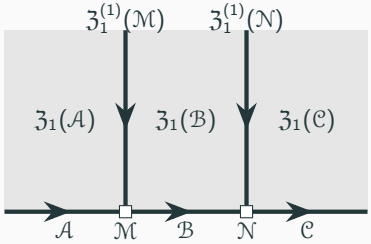
K.-Wen-Zheng:1502.01690 The boundary-bulk relation, i.e., the assignment:

$$\mathsf{A}^{n+1} \mapsto \mathsf{Z}(\mathsf{A})^{n+1}, \qquad \mathsf{M}^{n} \mapsto \mathsf{Z}^{(1)}(\mathsf{M})^{n+1},$$

gives a well-defined functor called *center functor* $\mathsf{Z} : \mathrm{TO}^{n+1}_{\mathrm{gp-wall}} \to {}^{\mathrm{af}}\mathrm{TO}^{n+2}_{\mathrm{af-gp-wall}}$.

$$\mathsf{Z}^{(1)}(\mathsf{N})^{n+1} \boxtimes_{\mathsf{Z}(\mathsf{B})} \mathsf{Z}^{(1)}(\mathsf{M})^{n+1} = \mathsf{Z}^{(1)}(\mathsf{M} \boxtimes_{\mathsf{B}} \mathsf{N})^{n+1}$$





Topological defects in A^{n+1} form a fusion n-category $\mathcal{A}$. The following assignment

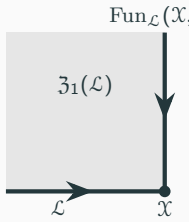
$$\mathcal{A} \mapsto \mathfrak{Z}_1(\mathcal{A}) := \mathcal{F}\mathrm{un}_{\mathcal{A}|\mathcal{A}}(\mathcal{A},\mathcal{A}), \qquad {}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}} \mapsto \mathfrak{Z}_1^{(1)}(\mathcal{M}) := \mathcal{F}\mathrm{un}_{\mathcal{A}|\mathcal{B}}(\mathcal{M},\mathcal{M})$$

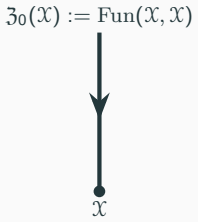
defines the so-called *center functor* $\mathfrak{Z}_1 : \mathcal{MF}us_n^{\mathrm{ind}} \to {}^{\mathrm{ndg}}\mathcal{BF}us_n^{\mathrm{cl}}$. K.-Wen-Zheng:1502.01690, 1507.00503

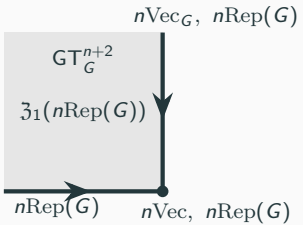
$$\begin{aligned} \mathcal{F}\mathrm{un}_{\mathcal{B}|\mathcal{C}}(\mathcal{N},\mathcal{N}) \boxtimes_{\mathfrak{Z}_1(\mathcal{B})} \mathcal{F}\mathrm{un}_{\mathcal{A}|\mathcal{B}}(\mathcal{M},\mathcal{M}) &\xrightarrow{\simeq} \mathcal{F}\mathrm{un}_{\mathcal{A}|\mathcal{C}}(\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N}, \mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N}) \\ g \boxtimes_{\mathfrak{Z}_1(\mathcal{B})} f \qquad &\mapsto \qquad f \boxtimes_{\mathcal{B}} g. \end{aligned}$$

Fundamental formula in computing a fusion or dimensional reduction:

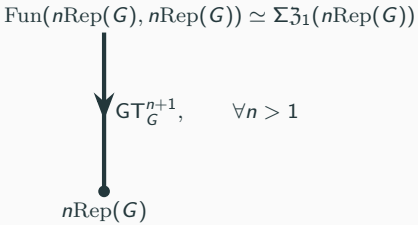
$$\mathrm{Fun}_{\mathcal{L}|\mathcal{M}}(\mathcal{X},\mathcal{X}) \boxtimes_{\mathfrak{Z}_1(\mathcal{M})} \mathrm{Fun}_{\mathcal{M}|\mathcal{N}}(\mathcal{Y},\mathcal{Y}) \simeq \mathrm{Fun}_{\mathcal{L}|\mathcal{N}}(\mathcal{X}\boxtimes_{\mathcal{M}}\mathcal{Y}, \mathcal{X}\boxtimes_{\mathcal{M}}\mathcal{Y})$$







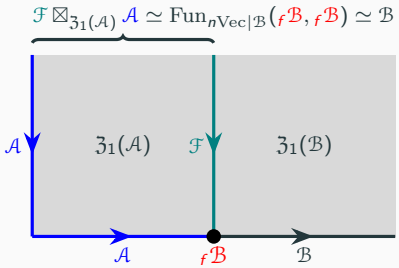




Morphisms are compatible with monoidal functors



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$$\{\text{monoidal functors } \mathcal{A} \to \mathcal{B}\} \quad \overset{1:1}{\Longleftrightarrow} \quad \{\text{“domain walls” between } \mathfrak{Z}_1(\mathcal{B}) \text{ and } \mathfrak{Z}_1(\mathcal{A}) \text{ s.t. } \mathcal{F} \boxtimes_{\mathfrak{Z}_1(\mathcal{A})} \mathcal{A} \simeq \mathcal{B}\}$$

$$(\mathcal{A} \xrightarrow{f} \mathcal{B}) \quad \longmapsto \quad \mathcal{F} := \mathrm{Fun}_{\mathcal{A}|\mathcal{B}}({}_f\mathcal{B}, {}_f\mathcal{B}),$$

$$\left(\mathcal{A} \xrightarrow{f} \mathcal{F} \boxtimes_{\mathfrak{Z}_1(\mathcal{A})} \mathcal{A} = \mathcal{B}\right) \quad \longmapsto \quad \mathcal{F}$$

Theorem ('Surjectivity' and 'fully-faithfulness' of the center functor K.-Zhang-Zhao-Zheng:2403.07813)

(1) $\mathfrak{Z}_1 : \mathcal{MF}us_{n-2}^{\mathrm{ind}} \to {}^{\mathrm{ndg}}\mathcal{BF}us_{n-2}^{\mathrm{cl}}$ *is 'essentially surjective' up to invertible separable n-categories, i.e.,*

0.1 $\mathcal{C} \in {}^{\mathrm{ndg}}\mathcal{BF}us_{n-2}^{\mathrm{cl}}$ *if and only if* $\Sigma^2\mathcal{C} \in (n+1)\mathrm{Vec}^{\times}$;

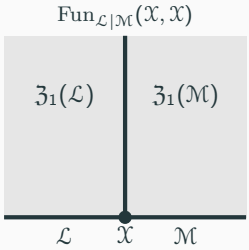
0.2 $\mathcal{C} \in \mathfrak{Z}_1(\mathcal{MF}us_{n-2}^{\mathrm{ind}})$ *if and only if* $\Sigma^2\mathcal{C} \simeq n\mathrm{Vec} \in (n+1)\mathrm{Vec}^{\times}$.

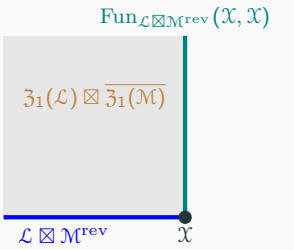
(2) $\mathfrak{Z}_1 : \mathcal{MF}us_{n-1}^{\mathrm{ind}} \to {}^{\mathrm{ndg}}\mathcal{BF}us_{n-1}^{\mathrm{cl}}$ *is 'full' up to invertible separable n-categories, i.e., for any 1-morphism* $\mathcal{K} : \mathfrak{Z}_1(\mathcal{A}) \to \mathfrak{Z}_1(\mathcal{B})$ *in* ${}^{\mathrm{ndg}}\mathcal{BF}us_{n-1}^{\mathrm{cl}}$ *and* $\mathcal{A}, \mathcal{B} \in \mathcal{MF}us_{n-1}^{\mathrm{ind}}$, *there exists a* $\mathcal{U} \in (n+1)\mathrm{Vec}^{\times}$ *and a non-zero* $\mathcal{M} \in \mathrm{BMod}_{\mathcal{A}|\mathcal{B}}((n+1)\mathrm{Vec})$ *such that* $\Sigma\mathcal{K} \simeq \Sigma\mathfrak{Z}_1^{(1)}(\mathcal{M}) \boxtimes \mathcal{U}$.

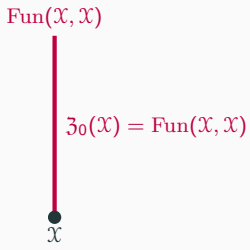
(3) $\mathfrak{Z}_1 : \mathcal{MF}us_{n}^{\mathrm{ind}} \to {}^{\mathrm{ndg}}\mathcal{BF}us_{n}^{\mathrm{cl}}$ *is 'faithful' up to invertible separable n-categories. That is, two 1-morphisms* ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}, {}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}}$ *for* $\mathcal{A}, \mathcal{B} \in \mathcal{MF}us_{n}^{\mathrm{ind}}$ *share the 'same' image, i.e.,* $\mathfrak{Z}_1^{(1)}(\mathcal{M}) \simeq \mathfrak{Z}_1^{(1)}(\mathcal{N})$, *if and only if there exists an* $\mathcal{V} \in (n+1)\mathrm{Vec}^{\times}$ *such that* $\mathcal{N} \simeq \mathcal{M} \boxtimes \mathcal{V}$.

In the special case $\mathcal{A} = \mathcal{B} = n\mathrm{Vec}$, *two separable n-categories* $\mathcal{M}$ *and* $\mathcal{N}$ *share the same center, i.e.,* $\mathfrak{Z}_0(\mathcal{M}) \simeq \mathfrak{Z}_0(\mathcal{N})$, *if and only if there exists a* $\mathcal{V} \in (n+1)\mathrm{Vec}^{\times}$ *such that* $\mathcal{N} \simeq \mathcal{M} \boxtimes \mathcal{V}$.

The center functor $\mathfrak{Z}_1 : \mathcal{MF}us_{1}^{\mathrm{ind}} \to {}^{\mathrm{ndg}}\mathcal{BF}us_{1}^{\mathrm{cl}}$ *is fully faithful.* K.-Zheng:1507.00503

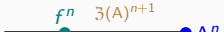




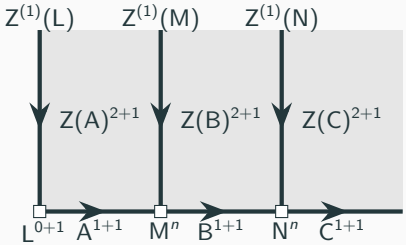


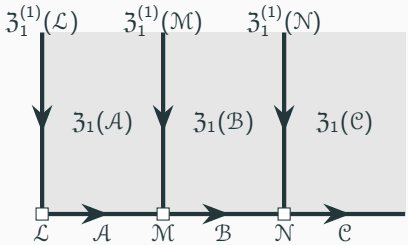
Theorem [K.-Zheng:1507.00503]: When $n=1$, this center functor is a monoidal equivalence.

Proof : ${}_{\mathcal{L}}\mathcal{X}_{\mathcal{M}} = {}_{\mathcal{L}\boxtimes\mathcal{M}^{\mathrm{rev}}}\mathcal{X}$ $\Longleftrightarrow$ a monodial functor : $\mathcal{L}\boxtimes\mathcal{M}^{\mathrm{rev}}\to\mathrm{Fun}(\mathcal{X},\mathcal{X})$

Recall : $(f:\mathsf{A}\to\mathsf{B})$ $=$ 

$\Longleftrightarrow$ a domain wall : $\mathrm{Fun}_{\mathcal{L}\boxtimes\mathcal{M}^{\mathrm{rev}}}(\mathcal{X},\mathcal{X})=\mathrm{Fun}_{\mathcal{L}|\mathcal{M}}(\mathcal{X},\mathcal{X})$.





When the center functor $\mathfrak{Z}_1 : \mathcal{MF}us_1^{\mathrm{ind}} \to {}^{\mathrm{ndg}}\mathcal{BF}us_1^{\mathrm{cl}}$ is fully faithful, we obtain immediately:

$\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent iff $\mathfrak{Z}_1(\mathcal{A}) \simeq \mathfrak{Z}_1(\mathcal{B})$ Müger:2001, Etingof-Nikshych-Ostrik:2008

$$\mathrm{BrPic}(\mathcal{A}) \xrightarrow[\text{Davydov-Müger-Nikshych-Ostrik:2013}]{\simeq} \mathrm{Aut}^{br}(\mathfrak{Z}_1(\mathcal{A}))$$

$$\{\text{ gapped boundaries of } \mathsf{Z(A)}^{2+1}\} \xrightarrow[\text{Kitaev-K.:2011}]{\simeq} \{\text{indecomposable right } \mathcal{A}\text{-modules}\}$$

This is the most important formula to compute sandwiches or dimensional reduction or factorization homology:

$$\begin{aligned} \mathcal{F}\mathrm{un}_{\mathcal{B}|\mathcal{C}}(\mathcal{N},\mathcal{N}) \boxtimes_{\mathfrak{Z}_1(\mathcal{B})} \mathcal{F}\mathrm{un}_{\mathcal{A}|\mathcal{B}}(\mathcal{M},\mathcal{M}) &\xrightarrow{\simeq} \mathcal{F}\mathrm{un}_{\mathcal{A}|\mathcal{C}}(\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N},\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N}) \\ g \boxtimes_{\mathfrak{Z}_1(\mathcal{B})} f \qquad &\mapsto \qquad f\boxtimes_{\mathcal{B}} g. \end{aligned} \tag{1}$$

Theorem (Ai-K.-Zheng:1607.08422)

Let Σ be a surface (space manifold) decorated by 2+1D topological orders for each 2-cells (i.e., $\mathcal{C}$), a gapped domain wall for each 1-cells (i.e., $\mathcal{M}$) and a defect junction for each 0-cells (i.e., $\mathcal{Q}$).

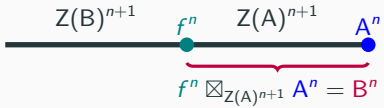
$$\int_{\Sigma} (\mathcal{C},\cdots;\mathcal{M},\cdots;\mathcal{Q},\cdots) \simeq (\mathrm{Vec}, u_{\Sigma}).$$

Generalizations and outlooks

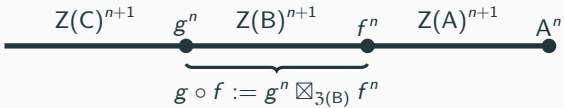
Unique Bulk Hypothesis/Principle [generalizing K.-Wen:1405.5858]

If (potentially anomalous) nD quantum phase A^n is scale invariant (e.g. QFT at RG fixed points), then it has a unique $n{+}1$D anomaly-free quantum phase as its *bulk*, denoted by $\mathrm{Z}(\mathrm{A}^n) = \mathrm{Z}(\mathrm{A})^{n+1}$.

A morphism $f : \mathrm{A}^n \to \mathrm{B}^n$ between two "nice" QFTs (or quantum liquids) A^n and B^n can be defined similarly as long as the fusion is well-defined (e.g., by cross-graining).



The composition of two morphisms $f : \mathrm{A}^n \to \mathrm{B}^n$ and $g : \mathrm{B}^n \to \mathrm{C}^n$ is defined similarly. The associativity of composition is needed but not required to hold on the nose (e.g., hold up to higher homotopies).



The proof of 'bulk=Z(boundary)' applies to all quantum liquids (i.e., fully dualizable QFTs) and more (for example, QFTs at RG fixed points).

1. It leads to the classification of all n+1D anomaly-free topological orders (up to invertible ones) by fusion n-categories with a trivial E_1-center, or equivalently, by braided fusion n-categories with a trivial E_2-center; K.-Wen:1405.5858, K.-Wen-Zheng:1502.01690, Johnson-Freyd:2003.06663, K.-Zheng:2011.02859,2107.03858

2. Moreover, 'bulk=Z(boundary)' applies to an n+1D topological order C^{n+1} with a gapless boundary A^n.

$$\frac{\text{gapped } \mathrm{C}^{n+1} = \mathrm{Z}(\mathrm{A})^{n+1}}{\text{gapless } \mathrm{A}^n}$$

   Since we already have a precise categorical description of C^{n+1} as a braided fusion n-categories $\mathcal{C}$ with a trivial E_2-center, then we can find the categorical description of a gapless boundary A^n by solving the mathematical equation $\mathfrak{Z}_1(?) = \mathcal{C}$. Since $\mathfrak{Z}_1(\mathcal{C}) \simeq \mathcal{C} \boxtimes \overline{\mathcal{C}}$, we can view $? \simeq \sqrt{\mathcal{C}}$. I will explain how to find this square-root in my next talk. References: K.-Zheng:1705.01087, 1905.04924, 1912.01760